III. TIME IN QUANTUM MECHANICS

In quantum mechanics the situation is essentially not different. The theory supposes a fixed, unquantized space-time background, the points of which are given by c-number coordinates ***x***, *t* . The space-time symmetry transformations are expressed in terms of these coordinates.

Dynamical variables of physical systems, on the other hand, are quantized: they are replaced by self-adjoint operators on a Hilbert space. All formulas of the preceding section remain valid if the Poisson-brackets are replaced by commutators according to { , } ® $(ih)^{-1}$ [,]. In particular, the canonical variables are replaced by operators satisfying the commutation relations:

(9) $[Q_k , P_l] = ih\, d_{kl}$; $[Q_k , Q_l] = [P_k , P_l] = 0$.

(In this section, symbols representing dynamical variables are supposed to be operators.)

Thus, for a point particle,

(10) $[q_i , p_j] = ih\, d_{ij}$; $[q_i , q_j] = [p_i , p_j] = 0$,

where (*i,j = x,y,z*) denote the Cartesian components of the position ***q*** and momentum ***p*** of the particle. These relations have the well-known representation where ***q*** is the multiplication operator and ***p*** the corresponding differentation operator. Both these operators are unbounded and have the full real axis as their spectrum. However, if the position wavefunctions are required to obey periodic boundary conditions the eigenvalues of ***p*** become discrete, and if the position wavefunctions are required to vanish at the endpoints of a finite interval (particle in a box) a self-adjoint momentum operator does not even exist. Corresponding statements hold for ***q*** . Similarly, since the wavefunctions of an angle variable must obey a periodic boundary condition, the eigenvalues of the corresponding angular momentum operator are discrete. Discrete energy eigenvalues are of course the hallmark of quantum mechanics. Nobody would conclude from these facts that something is totally wrong with the notions of position, momentum, angular momentum or energy in quantum mechanics. One should keep this in mind when we discuss quantum mechanical time-operators.

But first, let us point out some of the confusions which have established themselves in standard presentations of quantum mechanics as a result of mixing up ***q*** and ***x***.

Most texts on elementary quantum mechanics start by considering a single point particle. The particle position is commonly denoted ***x*** (instead of ***q***) and the time-dependent wave function is written y (***x**, t*). This notation is misleading in several ways. It gives the false impression that the wave function is just an ordinary wave in three-dimensional space, an impression which is reinforced by the usual discussions of double slit interference, quantum tunneling, et cetera. It seems that even von Neumann, in his quoted statement, has fallen victim to this notation. However, contrary to an ordinary field, like e.g. the electromagnetic field, y is a highly abstract entity, living in an abstract configuration space, carrying no energy and momentum but only *information* about the results of measurements. Furthermore, the notation suggests that ***x*** and *t* are quantities of the same kind which leads to the question why *t* , the universal time coordinate, is not an operator like ***x*** . The notation y (***q*** *, t*) for the wave function, where ***q*** denotes an eigenvalue of the position operator, would certainly have made this a less obvious question to ask.

Again, the idea, criticized in the previous section, that *t* can be seen as the canonical variable conjugate to the Hamiltonian, leads one to expect *t* to obey the canonical commutation relation [*t*, *H*] = *i* *h*. But if *t* is the universal time operator it should have continuous eigenvalues running from - ¥ to + ¥ and, from this, the same would follow for the eigenvalues of any *H*. But we know that discrete eigenvalues of *H* may occur. From this Pauli concluded: "…that the introduction of an operator *t* is basically forbidden and the time must necessarily be considered as an ordinary number ("c-number")….".

Thus, the 'unsolvable' problem of time in quantum mechanics has arisen.

Note that it is crucial for this argument that *t* is supposed to be a universal operator, valid for all systems: according to Pauli the introduction of such an operator is basically forbidden because *some* systems have discrete energy eigenvalues.

From our previous discussion it should be clear that the universal time coordinate *t* is the partner of the space coordinates ***x***. Neither the space coordinates nor the time coordinate is quantized in standard quantum mechanics. So, the above problem simply doesn't exist! If one is to look for a time *operator* in quantum mechanics one should not try to quantize the universal time coordinate but consider time-like (in the literal sense) dynamical variables of specific physical systems, i.e. clocks. Since a clock-variable is an ordinary dynamical variable quantization should not, in principle, be especially problematic. One must, however, be prepared to encounter the well-known quantum effects mentioned above: a dynamical system may have a continuous time-variable, or a discrete one or no time-variable at all. But this invalidates the notion of time in quantum mechanics as little as does, say, the discreteness of the energy eigenvalues of a system invalidates the notion of energy. Let us now turn to the quantum version of the clocks considered in section II.

IV. QUANTUM CLOCKS

In section II we have characterized ideal time-variables by their behavior under time translations, i.e. by relations (8). The analogous relations for the corresponding quantum mechanical operators are

(11) [h , *H*] = 0, [q , *H*] = *ih* .

The linear quantum clock

Relations (11) are satisfied by the quantum version of our simple linear clock if we take q

to be the multiplication operator, h = *-ih d/dq* and *H* = *h* . The operators q and *h* satisfy [q
,
h
] =
i
h
.
The spectrum of
q
,
h

and
H
is the whole real axis.
It is illuminating to compare this result with Pauli's argument above. Our q resembles the universal time parameter
t
as well as can be and our
H
has indeed continuous eigenvalues. But this does not imply that the Hamiltonians of
other
physical systems must have continuous eigenvalues also. It only means that in such systems time-operators either do not exist or cannot resemble
t
as closely as does our linear clock (although they can come pretty close, as we shall see). It is the supposed universality of the time-operator with is crucial for the validity of Pauli's argument.

But here another objection must be mentioned. Most physically interesting systems have Hamiltonians that are bounded from below. In such systems a time-operator having the real axis as its spectrum does not exist. Now, it is sometimes asserted that the Hamiltonian of real physical systems *must* be bounded from below in order to guarantee the stability of the system. The archetypical example is, of course, the Rutherford atom whose very instability gave rise to Bohr's atomic theory and to the development of modern quantum mechanics. However, the instability of the Rutherford atom is caused by its interaction with the electromagnetic field which allows it to dissipate its energy in the form of electromagnetic radiation. Without this interaction the atom would be stable because of the conservation of energy. Similarly, in quantum mechanics, the stationary states of the hydrogen atom are indeed stationary as long as the interaction of the atom with the electromagnetic field is not taken into account. So, for an isolated system, the demand that its Hamiltonian has a lower bound is not at all necessary. Thus, from the point of view of the quantum mechanical formalism our linear clock is a completely *bona fide* system.

It is ironic that the demand that *H* be bounded from below precludes the existence of an acceptable particle *position* operator in relativistic quantum mechanics (cf. section V).

The continuous cyclic quantum clock

This clock is characterized by an angle-variablef which will play the role of time-variable.

In quantum mechanics f is represented by an operator F . The Hilbert space in the angle representation consists of the square-integrable functions
f
of
f
on the interval [0,2
p
]. The operators of angle and angular momentum are represented by (
h
=1):

F *f*(f) = f *f*(f),

L f(f) = -*i f*(f).

The operator F is self-adjoint on the whole Hilbert space, whereas *L* is self-adjoint on the subspace of the square-integrable, differentiable functions satisfying
f
(0)
= *f*
(2
p
)
.
These operators have complete, orthonormal sets of generalized eigenstates
|
f
ñ

and |
m
ñ
:

F */f ñ* = f */f ñ* ,   á f |f ¢ ñ = d (f -*f ¢*)*,*

L |*mñ* = *m*|*mñ* ,   á *m*|*m¢ ñ* = d *m,m¢* ,

where the eigenvalue f runs through the interval [0,2p] and *m* = 0, ±1, ±2,…

In the f -representation the states */f ñ* and |*mñ* are represented by the wavefunctions á f |f ¢ ñ
=
d
(
f
-
f
¢
)
and
á
f
|
m
ñ
= (2
p

)
-½
e
-im
f
, respectively. The situation is very similar to that of the linear clock except for the fact that the interval on which the functions
f
(
f
)
are defined is finite. In particular, we have

(12) |*f* ñ = (2p)$^{-½}$ |*m*ñ ,

where the sum runs over all values of *m.*

The dynamics of the system is introduced by specifying a Hamiltonian; we put *H* = *w L*, where
w
is a constant frequency.

The operator of translations in time is *U*(*t*) = e^{-iHt}, and with the help of (12) we find:

U(*t*)|*f* ñ = e^{-iHt} |*f* ñ = (2p)$^{-½}$ å $e^{imf -imw t}$ |*m*ñ = |*f* -*w t*ñ which is precisely the behavior one expects of the hand of a clock: it rotates at constant angular velocity and after an arbitrarily short time an eigenstate *|f ñ* of the hand position goes over into an orthogonal state. Putting w = 1 we see that f plays the role of a time-variable: under a time translation it behaves exactly as *t* does.

The energy of this continuous clock is unbounded from below but the energy values are discrete. Returning once more to Pauli's argument we see that discrete energy eigenvalues do not rule out the existence of a decent time-operator, even though its spectrum is not the real axis.

Let us now see what happens if we restrict the energy of our clock.

The discrete cyclic quantum clock

Let us restrict the sum in (12) to values of *m* satisfying the condition *-l £ m £ l,* where *l* is a positive integer, and consider the 2*l+*1 orthogonal states

(13) |f $_{k\tilde{n}}$ = (2*l*+1)$^{-½}$ |*mñ* ,

where f $_{k}$ takes the values

$$\phi_k = 2\pi k/(2l+1),\ k = -l,\ldots,l.$$

We may now define a time-operator

$$Q \equiv \phi_k |\phi_k\rangle\langle\phi_k|.$$

The eigenvalues of Q are the $(2l+1)$ discrete times ϕ_k and if the system is in the eigenstate $|\phi_k\rangle$ at time t, it will be in the eigenstate $|\phi_{k+1}\rangle$ at time $t+2\pi/(2l+1)$, as may easily be verified by applying the evolution operator $U(t)$ to $|\phi_k\rangle$. This brings to mind the famous clock in the railway station which can only show discrete times! Note that by allowing l to increase we may approximate a continuous clock as closely as we wish.

The uncertainty principle

Time-operators, being ordinary operators, satisfy uncertainty relations with their canonical conjugates. Thus, for our linear clock it follows from [q ,h] = *ih* that q and h satisfy the usual Heisenberg uncertainty relation D q D h *з* ½ h . The case of the continuous cyclic clock is mathematically more complicated (just as is the case of angle and angular momentum) and we will not discuss it here. A very nice illustration of the uncertainty principle is provided by the discrete cyclic quantum clock. From (13) we see that in an eigenstate | f_k *ñ* of the time-operator all eigenstates of *L* appear with equal probability and the converse is also true. This means that if the value of f_k is maximally certain, the value of *L* is maximally uncertain, and conversely.

Note that in all these examples the conjugate operator of the time-operator coincides with the Hamiltonian. However, this need not be generally so; it is merely due to the simplicity of our examples (compare the case of a single particle where the conjugate momentum ***p*** coincides with the total momentum ***P***).

V. A REMARK ON POSITION IN RELATIVISTIC QUANTUM MECHANICS

Because of the dominant role particles and rigid bodies play in classical physics the notion

of the position of a physical system seemed unproblematic. This is still true in non-relativistic quantum mechanics, although we have seen that position-operators may have discrete eigenvalues (and may not even exist for special systems). However, in relativistic quantum mechanics the concept of a position-operator encounters serious problems. As could have been surmised from our remarks in section II, a point particle can mimic the behavior of a point in space but it cannot mimic the behavior of a point in spacetime. In a famous paper, T.D. Newton and E.P. Wigner showed that the required behavior of a position operator under space translations and rotations almost uniquely determines this operator. However, the resulting operator ***q*** is non-covariant and, due to its energy being positive, has the ugly property that a state which is an eigenstate of it at a given time (a so-called 'localized' state) will be spread out over all of space an infinitesimal time later. This result has given rise to an extensive literature on the feasibility of a localizable particle concept in relativistic quantum theory. In the case of a Dirac spin-½ particle the Newton-Wigner position operator turns out to be identical with the Foldy-Wouthuysen "mean position" operator. From our point of view, this case is particularly interesting for when the Dirac-equation was conceived in 1928 the space-part ***x*** of the four-vector appearing as the argument of Dirac's four-spinor 'wavefunction' y(***x***, *ct*), was identified with the position of the electron. This had the embarrassing consequence that the corresponding 'velocity' of the electron would always be found to be the velocity of light. It took twenty years before this problem was solved and the proper position-operator ***q*** was identified. Here again, the notation ***x*** for both the particle position and the space-coordinate certainly has obscured the issue.

In relativistic quantum field theory neither the position of a particle nor the concept of a clock-variable play a role. There, the basic quantity is the operator field f (***x***,*t*) which is parametrized by the c-number coordinates of spacetime points.

VI. CONCLUSION

When looking for a time-operator in quantum mechanics, a distinction must be made between the universal time coordinate *t*, a c-number like the space-coordinates, and dynamical time-variables of physical systems situated *in* space-time. Time-variables stand in a particularly simple relation to *t* and do exist in specific physical systems: clocks. In the quantum formalism position- and time-variables are not treated differently. Much of the confusion about time in quantum mechanics has been caused by not making a proper distinction, in classical as well as in quantum physics, between position-variables of particles and coordinates of points of space. Dynamical position- and time-variables of material systems are essentially non-covariant quantities. The demands of relativistic covariance are so stringent in quantum mechanics that no concept of a point particle can meet them, rendering the whole discussion about dynamical time- and position-variables somewhat obsolete. The quantum field seems to be the more fruitful concept to incorporate relativistic covariance in quantum physics.

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