TIME IN QUANTUM MECHANICS

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Abstract

Time is often said to play in quantum mechanics an essentially different role from position: whereas position is represented by a Hermitian operator, time is represented by a c-number. This discrepancy has been found puzzling and has given rise to a vast literature and many efforts at a solution. In this paper it is argued that the discrepancy is only apparent and that there is nothing in the formalism of (standard) quantum mechanics that forces us to treat position and time differently. The apparent problem is caused by the dominant role point particles play in physics and can be traced back to classical mechanics.

I. **INTRODUCTION**

From the early days, the role time plays in quantum mechanics has caused worries. Thus, von Neumann, in his famous book, complains:

"First of all we must admit that this objection points at an essential weakness which is, in fact, the chief weakness of quantum mechanics: its non-relativistic character, which distinguishes the time t from the three space coordinates x,y,z, and presupposes an objective simultaneity concept. In fact, while all other quantities (especially those x,y,z closely connected with t by the Lorentz transformation) are represented by operators, there corresponds to the time an ordinary number-parameter t, just as in classical mechanics." It is true, of course, that elementary quantum mechanics is not relativistic, but it is not true that the three space coordinates are operators in quantum mechanics.

Seventy years later little seems to have changed when we read: "Moreover, space and time are treated very differently in quantum mechanics. The spatial co-ordinates are operators, whereas time is a parameter …"

Most textbooks of the intervening period tell us that time is exceptional in quantum mechanics and many efforts to deal with this problem have appeared in the literature.

In the following I will show that time does not pose a special problem for quantum mechanics.

II. **TIME IN CLASSICAL MECHANICS**

Quantum mechanics was modeled on classical Hamiltonian mechanics. In Hamiltonian mechanics a physical system is described by *N* pairs of canonical conjugate dynamical variables, Q_k and P_k, which satisfy the Poisson-bracket relations:

$$\{Q_k, P_l\} = d_{kl}\,;\ \{Q_k, Q_l\} = \{P_k, P_l\} = 0.$$

These variables define a point of the *2N*-dimensional so-called 'phase space' of the system. The time evolution of the system is generated by the Hamiltonian, a function of the canonical variables, $H = H(Q_k, P_k)$:

$dQ_k/dt = \{Q_k , H\}$, $dP_k/dt = \{P_k , H\}$.

(We assume that H does not explicitly depend on time.)

The Q_k and P_k are generalized variables; they need not be positions and momenta, but may be angles, angular momenta, et cetera. However, if the system is a system of point particles the canonical variables are usually taken to be the positions $\boldsymbol{q}_n$ and momenta $\boldsymbol{p}_n$ of the particles (three-vectors are in bold type and the subscript denotes the n-th particle). Let us consider the relation of this scheme with space and time.

In all of physics, with the exception of Einstein's Theory of Gravity (General Relativity), physical systems are supposed to be situated in a three-dimensional Euclidean space. The points of this space will be given by Cartesian coordinates $\boldsymbol{x} = (x,y,z)$. Together with the time parameter t they form the coordinates of a continuous, independently given, space-time background. How the existence of this space and time is to be justified is an important and difficult problem into which I will not enter; I just take this assumption as belonging to the standard formulation of classical and quantum mechanics and of special relativity.

The (3+1)-dimensional space-time must be sharply distinguished from the $2N$-dimensional phase space of the system, and the space-time coordinates ($\boldsymbol{x}$, t) must be sharply distinguished from the dynamical variables (Q_k, P_k) characterizing material systems *in* space-time. In particular, the position variable $\boldsymbol{q}$ of a point particle must be distinguished from the coordinate $\boldsymbol{x}$ of the space-point the particle occupies, although we have the numerical relation:

$q_x = x$, $q_y = y$, $q_z = z$.

A point particle is a material system having a mass, a position, a velocity, an acceleration, while ***x*** is the coordinate of a fixed point of empty space. We will see that mixing up ***q*** and ***x*** is at the very root of the problem of *time* in quantum mechanics!

A vital role is played in physics by the symmetries space and time are supposed to possess. It is assumed that three-dimensional space is isotropic (rotation symmetric) and homogeneous (translation symmetric) and that there is translation symmetry in time. In special relativity the space-time symmetry is enlarged by Lorentz transformations which mix ***x*** and *t*, transforming them as the components of a four-vector. (In a relativistic context I shall write 'spacetime' in stead of space-time.)

For the following it is important to note that individual physical systems *in* space-time need not show these symmetries; only the physical laws, that is the totality of physically allowed situations and processes, must show them. A given physical system need not be rotation invariant, and a position variable of a physical system need not be part of a four-vector.

The generators of translations in space and time are the total momentum ***P*** and the total energy *H*, respectively. The generator of space rotations is the total angular momentum ***J***. We shall in particular be interested in the behavior of dynamical variables under translations in

time
and space. For an infinitesimal translation
d
t
in time we have:

(1) $d Q_k = \{Q_k , H\} dt$, $d \mathbf{P}_k = \{P_k , H\} dt$,

and for an infinitesimal translation $d\mathbf{a}$ in space:

(2) $d Q_k = \{Q_k , \mathbf{P}\}.d\mathbf{a}$, $d P_k = \{P_k , \mathbf{P}\}.d\mathbf{a}$.

At this point one may wonder why the Hamiltonian, the generator of time translations, i.e. of the time evolution of the system, is so much more prominent in classical mechanics than is the total momentum, the generator of translations in space. The reason for this is that the dynamical variables of the systems which are traditionally studied in classical mechanics, namely particles and rigid bodies, transform trivially under space translations. For example, for a system of particles, a space translation $\mathbf{a}$

(3) $\mathbf{x} \rightarrow \mathbf{x} + \mathbf{a}$, $t \rightarrow t$,

induces the simple transformation

(4) $\mathbf{q}_n \rightarrow \mathbf{q}_n + \mathbf{a}$, $\mathbf{p}_n \rightarrow \mathbf{p}_n$,

of the canonical variables. The infinitesimal form of (4) is $d\mathbf{q}_n = d\mathbf{a}$; $d\mathbf{p}_n = 0$ and, comparing this with (2), we find the simple relations:

$\{q_{n,i}\ , P_j\} = d_{ij}\ ,\ \ \{p_{n,i}\ , P_j\} = 0\ ;\ (i, j = x,y,z),$

with the obvious solution ***P*** = S ***p*** ***n***.

In case there is only one particle and one space dimension, this becomes:

(5) $\{q\ , P\} = 1, \{p\ , P\} = 0.$

The simplicity of behavior under space translations shown by a point particle is not a general feature of the theory. If the physical system is a *field*, like e.g. an electromagnetic field strength or the density distribution in a fluid, the effect of a translation in space may be as non-trivial as the effect of a translation in time and *H* and ***P*** are equally important. In relativity theory, *H* and ***P*** are combined into the components of a four-vector.

The great similarity between the behavior under space translations (and rotations) of the position ***q*** of a point particle and the coordinate ***x*** of a point of three-dimensional space obscures the conceptual difference between the two, and the wide-spread use of the notation ***x*** for the position of a particle has greatly added to this confusion. In many discussions in classical mechanics an explicit distinction between ***x*** and ***q*** is even never made. Although ignoring the distinction may be innocent when it is sufficiently clear what is meant, it has, nevertheless, caused important misunderstandings of a general kind of which we will see examples here and in the following section.

In relativity theory the coordinates ***x,*** *t* transform as the components of a Lorentz four-vector. This has led people into believing that the position ***q*** of a particle should also be part of a four-vector with the time coordinate t as the fourth component. But $\boldsymbol{q}$ is a dynamical variable belonging to a material system whereas t is a universal spacetime coordinate. No one would think of adding t to the position variables of an arbitrary physical system, say a rigid body, to form a four-vector. It is only the great resemblance of a point particle to a space point which has misled people in this case. Already the case of a system consisting of several particles should be eye-opening here: in this case one would have to combine the very same t with *all* position variables.

As remarked above, a symmetry of spacetime does not imply the same symmetry of every physical system in spacetime. The point particle is a case in point. It simply does not possess a dynamical variable which may be combined with its position variable to form a four-vector. The position of a point particle is an essentially non-covariant concept. (Its momentum and energy, on the other hand, do form a four-vector.)

Another confusion, possibly related to the above one, lies at the root of efforts to include the time parameter t in the set of canonical variables as the partner conjugate to H. Again, since H and t differ conceptually in the same way as do $\boldsymbol{q}$ and t, such efforts are misconceived. In fact, H, being a given function of the original canonical variables, is not an independent canonical variable. Such a manoeuvre, therefore, implies a severe departure from the original scheme. Had the roles of $\boldsymbol{x}$ and $\boldsymbol{P}$**,** the analogues of t

and
H
,
been more clearly recognized in classical mechanics, the temptation to add
t
to the canonical variables, while leaving
x
alone, would probably not have arisen.

Time as a dynamical variable

The above served to stress the conceptual difference between the space-time coordinates and the dynamical variables of physical systems in space-time. In particular, the universal time coordinate *t* should not be mixed with dynamical position variables. But do physical systems exist that have a dynamical variable which resembles the time coordinate *t* in the same way as the position variable
q
of a point particle resembles the space coordinate
x
? The answer is yes! Such systems are
clocks
. A clock stands, ideally, in the same simple relation to the universal time coordinate
t
as a point particle stands to the universal space coordinate
x
. We may generally define an ideal clock as a physical system which has a dynamical variable which behaves under time translations in the same way as the time coordinate
t
. Such a variable, which we shall call a "clock-variable" or, more generally, a "time-variable", may be a pointer position or an angle or even a momentum. Just as a position variable indicates the position of a system in space, a clock-variable indicates the 'position' of a system in time. The closest analogue to (3) and (4) would be a linear clock-variable
q
with a conjugate momentum
h
such that a time translation

(6) ***x*** ® ***x*** , *t* ® *t + t*

induces the simple transformation

(7) *h* ® h , *q* ® q +t .

Comparing the infinitesimal form of this transformation with (1), we find:

(8) {h , *H*} = 0, {q , *H*} = 1,

which is the analog of (5) for time-variables.

The simplest solution of (8) is *H*(q ,h) = *h* . This is analogous to the case of a single particle where the total momentum *P*(*q,p*) coincides with the momentum *p* . The equation of motion *d/dt* *q* = { q , *H* } = 1 has the solution q = *t* which is the analogue of the relation *q* = *x* for a point particle.

A model of a linear clock is provided by a particle moving in a constant force field. The momentum of the particle is a linear function of *t* and furnishes a time-variable. More precisely, starting from the Hamiltonian *H*(*q*, *p*) = $p^2/2m$ - *Kq* of the particle in the field *K*, we go over to the variables q, h by the canonical transformation:

$q \circledR q = p/K$, $p \circledR h = H = p^2/2m - Kq$.

Then, $\{q, p\} = \{q, h\} = 1$, and $H(q, h) = h$. It follows that $d/dt\, q = 1$.

Note that *H* is unbounded: it may take on any real value.

Actual clocks are not ideal in the sense of (8); in fact, most real clocks like e.g. a pendulum clock or a quartz clock are not even continuous indicators of time. What the example purports to show is that the Hamiltonian formalism allows for the existence of systems satisfying (8), playing the same role with respect to *H* and *t* as point particles do with respect to *P* and *x*.

The cyclic clock-variable corresponding to our linear clock is an angle variable f with conjugate momentum *L* and Hamiltonian *H*(f, *L*) = *L*. Here f = *t* (mod 2p). We shall come back to these examples in section IV.

We conclude that in classical physics a sharp distinction must be made between the universal space-time coordinates and the dynamical variables of specific physical systems situated *in* space-time. Particles and clocks are physical systems having dynamical variables which behave in much the same way as the space and time coordinates, respectively, and may thus serve to indicate the 'position' of the system in space and in time. Point particles and clocks are non-covariant concepts. If one is to look for physical systems which transform covariantly under relativistic space-time transformations one must consider *fields*.